

Large Tensor-to-Scalar Ratio in Small-Field Inflation

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We show that density perturbations seeded by the inflaton can be suppressed when having additional light degrees of freedom contributing to the production of perturbations. The inflaton fluctuations affect the light field dynamics by modulating the length of the inflationary period, hence produce additional density perturbations in the post-inflationary era. Such perturbations can cancel those generated during inflation as both originate from the same inflaton fluctuations. This allows production of large gravitational waves from small-field inflation, which is normally forbidden by the Lyth bound on the inflaton field excursion. We also find that the field bound is taken over by the light scalar when the inflaton-induced perturbations are suppressed, thus present a generalized form of the Lyth bound that applies to the total field space.

Introduction.— The origin of cosmic structure, or primordial density perturbations, is an important issue in cosmology. One of the key observables is the gravitational waves which are usually quantified by the tensor-to-scalar ratio r . In standard inflationary scenarios, there exists the so-called Lyth bound [1] which constrains r in terms of the inflaton field excursion as $r \lesssim 0.01(\Delta\phi/M_p)^2$ with M_p being the reduced Planck mass. This can be rephrased as small-field (i.e. sub-Planckian) inflation models only producing tiny r . The bound is due to the inflaton with super-horizon field fluctuations necessarily producing density perturbations, and it should be noted that the bound can become more restrictive, but not alleviated, by non-slow-roll inflation (without superluminal modes) [2], or by simply adding extra sources for the density perturbations [3].

In this letter we point out that this general belief is not necessarily true when there are additional light fields producing density perturbations such as in the curvaton scenario [4] or modulated reheating [5]. The basic idea can be explained as follows: The inflaton field fluctuations source density perturbations by giving slightly longer/shorter inflationary periods among different patches of the universe. This also affects the dynamics of the light fields by allowing more/less time to roll along their potentials during inflation. Therefore the inflaton field fluctuations induce fluctuations of the light fields as well, leading to further generation of density perturbations in the post-inflationary era. The inflaton-induced perturbations generated during and after inflation (note that both are seeded by the same inflaton fluctuations) can cancel each other, then one can evade the Lyth bound for the inflaton. We demonstrate that density perturbations from the inflaton fluctuations can actually be suppressed, and show that large tensor-to-scalar ratio can be obtained even in small-field inflation models with the aid of an additional light field generating perturbations. However the total perturbations remain finite, hence in order to evade the field range bound for the in-

flaton, the other field instead has to take over the large field excursion. In this sense the Lyth bound is shown to apply to the total field space of the inflaton and the light field. After deriving the generalized form of the Lyth bound, we present an explicit scenario where perturbations from the inflaton are inevitably suppressed.

Lyth bound for the inflaton and a light scalar.— Let us study field range bounds when there is an extra light scalar σ besides the inflaton ϕ producing density perturbations as in, for e.g., the curvaton or modulated reheating mechanisms. We assume that the σ field has negligibly tiny energy density during inflation and also that its dynamics has little effect on the inflationary expansion, but contributes to the production of density perturbations in the post-inflationary era. The fields ϕ and σ interact with each other only via gravity.

In order to compute the density perturbations seeded by the fields' fluctuations, we use the $\delta\mathcal{N}$ -formalism and study the evolution of the universe which is uniquely determined by a set of field values (ϕ, σ) at an arbitrary time when the fields follow attractor solutions. The density perturbations are obtained by computing the fluctuations in the e-folding number:

$$\mathcal{N} = \int_{t_*}^{t_{\text{end}}} H dt + \int_{t_{\text{end}}}^{t_f} H dt \equiv \mathcal{N}_a + \mathcal{N}_b, \quad (1)$$

where the subscript “*” denotes quantities when the CMB scale exits the horizon, “end” at the end of inflation, and “f” at the final uniform density hypersurface after which no further $\delta\mathcal{N}$ is produced. \mathcal{N} is split into \mathcal{N}_a and \mathcal{N}_b at t_{end} . Considering the fields to follow attractor solutions at least until the end of inflation, then

$$\frac{\partial \mathcal{N}}{\partial \phi_*} = \frac{\partial \mathcal{N}_a}{\partial \phi_*} + \frac{\partial \phi_{\text{end}}}{\partial \phi_*} \frac{\partial \mathcal{N}_b}{\partial \phi_{\text{end}}} + \frac{\partial \sigma_{\text{end}}}{\partial \phi_*} \frac{\partial \mathcal{N}_b}{\partial \sigma_{\text{end}}}, \quad (2)$$

$$\frac{\partial \mathcal{N}}{\partial \sigma_*} = \frac{\partial \sigma_{\text{end}}}{\partial \sigma_*} \frac{\partial \mathcal{N}_b}{\partial \sigma_{\text{end}}}. \quad (3)$$

Note that \mathcal{N}_a and ϕ_{end} are independent of σ_* since the inflationary expansion is governed by the inflaton. Here-

after we suppose that the end of inflation is set by a constant inflaton field value ϕ_{end} . (Hence we do not consider inhomogeneous end of inflation [6], but the generalization to such cases is straightforward.) This yields

$$\frac{\partial \mathcal{N}_a}{\partial \phi_*} = \frac{\partial}{\partial \phi_*} \int_{\phi_*}^{\phi_{\text{end}}} \frac{H}{\dot{\phi}} d\phi = -\frac{H_*}{\dot{\phi}_*}, \quad (4)$$

where an overdot represents a time derivative.

Considering the field σ to be light during inflation and that it slow-rolls along its potential $V(\sigma)$, i.e. $3H\dot{\sigma} = -V'(\sigma)$ with a prime denoting a σ -derivative, one obtains

$$\int_{\sigma_*}^{\sigma_{\text{end}}} \frac{d\sigma}{V'(\sigma)} = - \int_{\phi_*}^{\phi_{\text{end}}} \frac{d\phi}{3H\dot{\phi}}. \quad (5)$$

Partially differentiating both sides by ϕ_* gives

$$\frac{\partial \sigma_{\text{end}}}{\partial \phi_*} = \frac{V'(\sigma_{\text{end}})}{3H_* \dot{\phi}_*}, \quad (6)$$

while differentiating with σ_* (note that the right hand side of (5) is independent of σ_*) yields

$$\frac{\partial \sigma_{\text{end}}}{\partial \sigma_*} = \frac{V'(\sigma_{\text{end}})}{V'(\sigma_*)}. \quad (7)$$

Hence by combining the above results, one arrives at

$$\frac{\partial \mathcal{N}}{\partial \phi_*} = -\frac{H_*}{\dot{\phi}_*}(1-\kappa), \quad \text{where} \quad \kappa \equiv \frac{V'(\sigma_*)}{3H_*^2} \frac{\partial \mathcal{N}}{\partial \sigma_*}. \quad (8)$$

This κ represents the effect of $\delta\phi$ modulating the field value of σ at the end of inflation, and thus further generating density perturbations in the post-inflationary era.

Let us suppose Gaussian field fluctuations with power spectra $\mathcal{P}_{\delta\phi_*}^{1/2} = \mathcal{P}_{\delta\sigma_*}^{1/2} = H_*/2\pi$, with no correlations between the two. Then the total density perturbations are written as a sum of contributions from each field,

$$\mathcal{P}_\zeta = \mathcal{P}_{\zeta\phi} + \mathcal{P}_{\zeta\sigma} = \left(\frac{\partial \mathcal{N}}{\partial \phi_*} \frac{H_*}{2\pi} \right)^2 + \left(\frac{\partial \mathcal{N}}{\partial \sigma_*} \frac{H_*}{2\pi} \right)^2. \quad (9)$$

The amplitude of tensor perturbations (gravitational waves) \mathcal{P}_T is set only by the inflation scale, so in terms of the tensor-to-scalar ratio r defined as

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \quad \text{with} \quad \mathcal{P}_T = \frac{2H_*^2}{\pi^2 M_p^2}, \quad (10)$$

we arrive at the generalized form of the Lyth bound:

$$\left(\frac{1}{M_p} \frac{\dot{\phi}_*}{H_*} \right)^2 = \frac{(1-\kappa)^2}{8} \frac{\mathcal{P}_\zeta}{\mathcal{P}_{\zeta\phi}} r \geq \frac{(1-\kappa)^2}{8} r, \quad (11)$$

$$\left(\frac{1}{M_p} \frac{\dot{\sigma}_*}{H_*} \right)^2 = \frac{\kappa^2}{8} \frac{\mathcal{P}_\zeta}{\mathcal{P}_{\zeta\sigma}} r \geq \frac{\kappa^2}{8} r. \quad (12)$$

The left hand sides represent the field excursions during one Hubble time at around when the CMB scale exits the

horizon. The total field excursions can be estimated by multiplying the expressions by the total number of inflationary e-folds [11]. One recovers the familiar result for a single inflaton for $\kappa = 0$, i.e. when there is no additional degree of freedom generating density perturbations.

On the other hand when $\kappa \approx 1$, density perturbations sourced from the inflaton fluctuations are suppressed (cf. (8)), and the lower bound on the inflaton field excursion (11) becomes tiny. However it should be noted that in such cases, the required field range for σ (12) increases instead. Thus one sees that the familiar form of the Lyth bound now applies to the total field space of ϕ and σ .

One can further check that the total power spectrum (9) is nearly scale-invariant for $\kappa^2 r \lesssim 0.1$, given that both ϕ and σ slow-roll during inflation.

Before ending this section, we should also remark on the variation of the energy density of the field σ :

$$\left| \frac{1}{3M_p^2 H_*^2} \frac{\dot{V}(\sigma_*)}{H_*} \right| = \frac{\kappa^2}{8} \frac{\mathcal{P}_\zeta}{\mathcal{P}_{\zeta\sigma}} r > \frac{\kappa^2}{8} r, \quad (13)$$

which is similar to the field bound (12) except for the power of the left hand side. One sees that $\kappa^2 r$ as large as ~ 0.1 requires the variation $\Delta V(\sigma)$ during inflation to be comparable to the inflation scale itself. In such a case σ is also regarded as the inflaton in the sense that its dynamics directly affects the inflationary expansion, then the discussions in this section need to be modified accordingly.

An example: early oscillating curvaton.— As an example, we present a curvaton scenario where the effective potential forces the curvaton to start oscillating during inflation (cf. Fig. 1). In such cases, inflaton-induced perturbations are inevitably suppressed given that the curvaton dominates the universe before decaying away. The computations in the previous section do not strictly apply here since the curvaton ceases to slow-roll during inflation (cf. discussions around (5)), however the basic assumptions and arguments stay the same [12].

We consider a curvaton σ whose potential $V(\sigma)$ allows the field to slow-roll while the CMB scales exit the horizon, but starts the curvaton oscillation before the end of inflation when σ approaches σ_{osc} . Here, σ_{osc} is a constant

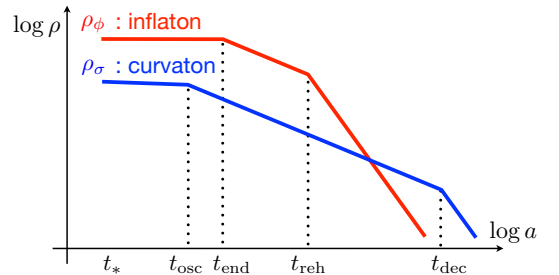


FIG. 1: Schematic of the time variation of energy densities.

field value that is basically set by $V(\sigma)$ [13]. An explicit example of such potential is given later in (24). $V(\sigma)$ is assumed to be well approximated by a quadratic around its minimum, and thus the energy density of the oscillating curvaton redshifts as nonrelativistic matter [14].

Inflation ends when ϕ approaches a constant field value ϕ_{end} , after which the inflaton also behaves as matter until it decays into radiation (reheating), and eventually the curvaton also decays away and thermalize with the inflaton decay products. We assume the fields to suddenly decay when H is equal to their constant decay rates Γ_ϕ and Γ_σ . The curvaton energy density is negligibly tiny during inflation, hence also until reheating.

Taking the final hypersurface as when the curvaton decays, let us break up the e-folding number as (the subscript “osc” denotes the onset of curvaton oscillation, “reh” the inflaton decay, and “dec” the curvaton decay)

$$\mathcal{N} = \left(\int_{t_*}^{t_{\text{osc}}} + \int_{t_{\text{osc}}}^{t_{\text{end}}} + \int_{t_{\text{end}}}^{t_{\text{reh}}} + \int_{t_{\text{reh}}}^{t_{\text{dec}}} \right) H dt \quad (14)$$

$$\equiv \mathcal{N}_a + \mathcal{N}_b + \mathcal{N}_c + \mathcal{N}_d.$$

Then we have

$$\frac{\partial}{\partial \phi_*} (\mathcal{N}_a + \mathcal{N}_b) = -\frac{H_*}{\dot{\phi}_*}, \quad \frac{\partial}{\partial \sigma_*} (\mathcal{N}_a + \mathcal{N}_b) = 0. \quad (15)$$

Since H_{end} is set by the constant ϕ_{end} , the e-folding number $\mathcal{N}_c = \ln(H_{\text{end}}/\Gamma_\phi)^{2/3}$ is independent of ϕ_* or σ_* . Denoting the energy density of radiation produced from the inflaton decay by ρ_r , then $\dot{\rho}_r = -4H\rho_r$ gives

$$\mathcal{N}_d = \frac{1}{4} \ln \frac{\rho_{r \text{ reh}}}{\rho_{r \text{ dec}}} = \frac{1}{4} \ln \frac{3M_p^2 \Gamma_\phi^2}{3M_p^2 \Gamma_\sigma^2 - \rho_{\sigma \text{ dec}}}, \quad (16)$$

with the curvaton’s energy density upon its decay

$$\rho_{\sigma \text{ dec}} = V(\sigma_{\text{osc}}) \exp \{-3(\mathcal{N}_b + \mathcal{N}_c + \mathcal{N}_d)\}. \quad (17)$$

Partially differentiating both sides of (16) yields

$$\frac{\partial \mathcal{N}_d}{\partial \phi_*} = -\frac{3\hat{r}}{4+3\hat{r}} \frac{\partial \mathcal{N}_b}{\partial \phi_*}, \quad (18)$$

where \hat{r} is the energy density ratio at curvaton decay,

$$\hat{r} \equiv \left. \frac{\rho_\sigma}{\rho_r} \right|_{\text{dec}}. \quad (19)$$

We note that (18) also holds for $\partial/\partial \sigma_*$.

Now let us suppose that inflation is a small-field type such that the time variation of H is very small [15], and adopt a constant scale H_{inf} to represent the inflationary Hubble parameter, i.e. $H \simeq H_{\text{inf}}$. (This treatment will induce errors of $\sim |\dot{H}/H^2|$.) Then the slow-roll approximation for the curvaton gives

$$\mathcal{N}_a \simeq \int_{\sigma_{\text{osc}}}^{\sigma_*} d\sigma \frac{3H_{\text{inf}}^2}{V'(\sigma)}, \quad (20)$$

which leads to

$$\frac{\partial \mathcal{N}_a}{\partial \phi_*} \simeq \mathcal{O}\left(\frac{\dot{H}}{H^2}\right) \times \frac{H_*}{\dot{\phi}_*}, \quad \frac{\partial \mathcal{N}_a}{\partial \sigma_*} \simeq \frac{3H_{\text{inf}}^2}{V'(\sigma_*)}. \quad (21)$$

The ϕ_* -dependence drops out of the approximation (20), hence we have given an order of magnitude estimation for $\partial \mathcal{N}_a/\partial \phi_*$. Therefore, by combining the above equations, we arrive at the final results

$$\frac{\partial \mathcal{N}}{\partial \phi_*} \simeq -\frac{4}{4+3\hat{r}} \frac{H_*}{\dot{\phi}_*}, \quad (22)$$

$$\frac{\partial \mathcal{N}}{\partial \sigma_*} \simeq \frac{3\hat{r}}{4+3\hat{r}} \frac{3H_{\text{inf}}^2}{V'(\sigma_*)} \simeq -\frac{3\hat{r}}{4+3\hat{r}} \frac{H_*}{\dot{\sigma}_*} \quad (23)$$

We repeat that these expressions contain errors of order $|\dot{H}/H^2|$ during inflation (see also discussions at the end of this section) [16][17]. Here one clearly sees that perturbations from the inflaton are highly suppressed if the curvaton dominates the universe before decaying, i.e. $\hat{r} \gg 1$. The suppression of the inflaton-induced perturbations is due to the curvaton “not knowing” about the length of the inflationary period: The curvaton field evolves and then starts to oscillate during inflation almost independently of the inflaton dynamics, especially for small-field inflation. Hence its energy density is almost uniquely determined by the e-folding number, and as the curvaton dominates the universe it dilutes away the consequences of the inflaton fluctuations.

The field bounds (11) and (12) now apply with $\kappa \simeq 3\hat{r}/(4+3\hat{r})$. Since (12) no longer holds after the curvaton starts oscillating, one may think that the total field excursion is reduced (in a similar fashion as in Footnote [11]). However an earlier onset of the oscillations (i.e. larger \mathcal{N}_b) quickly decreases ρ_σ and thus delays curvaton domination. One can check that \mathcal{N}_b cannot be much bigger than 10 for the curvaton to dominate the universe before the Big Bang Nucleosynthesis, hence the total σ -range is not greatly reduced.

We have carried out analytic and numerical calculations for a curvaton potential of the form

$$V(\sigma) = \Lambda^4 \left(\frac{\sigma}{f} \right)^2 \left[1 + \left(\frac{\sigma}{f} \right)^2 \right]^{-1/2}, \quad (24)$$

which is quadratic around the origin but approaches a linear potential for $|\sigma| \gg f$. The curvaton initially slow-rolls along the linear part, and then starts oscillating as it approaches $|\sigma| \sim f$ [18]. The inflationary scale is fixed to $H_{\text{inf}} \approx 8.4 \times 10^{12} \text{ GeV}$ and the curvaton potential tilt to $(\Lambda^4/f)^{1/3} \approx 1.8 \times 10^{14} \text{ GeV}$ so that when $\hat{r} \gg 1$, the curvaton-induced perturbations take the WMAP value $\mathcal{P}_{\zeta\sigma} \approx 2.4 \times 10^{-9}$ [10] and the tensor-to-scalar ratio is $r = 0.001$. (The explicit values of Λ and f are irrelevant here, as long as the effective mass $m_\sigma = \sqrt{2}\Lambda^2/f$ at the minimum is sufficiently larger than H_{inf} .) We set the inflaton to possess a small-field type potential $U(\phi)$ with

an almost constant tilt $\epsilon \equiv M_p^2 U'^2 / 2U^2 \approx 10^{-6}$, with field excursion $|\phi_* - \phi_{\text{end}}| \approx 0.071 M_p$ to support about 50 e-folds between t_* and t_{end} . On the other hand, the initial curvaton field value is taken to be of order the Planck scale $\sigma_* \sim 0.5 M_p$, cf. (12). The resulting density perturbations are plotted in Fig. 2 in terms of the energy ratio \hat{r} , which varies depending on the decay rates Γ_ϕ and Γ_σ . The analytic results from (22) and (23) are shown as lines (red dashed: $\mathcal{P}_{\zeta\phi}$, blue dotted: $\mathcal{P}_{\zeta\sigma}$, black solid: total \mathcal{P}_ζ), while the dots represent numerical results (red: $\mathcal{P}_{\zeta\phi}$, blue: $\mathcal{P}_{\zeta\sigma}$) obtained by solving the fields' equations of motion and computing $\delta\mathcal{N}$ by varying the initial field configuration. The analytic and numerical results agree well, and one clearly sees that the inflaton-induced perturbations are suppressed for $\hat{r} \gg 1$, allowing large tensor-to-scalar ratio with a small inflaton field range. Note especially that r is now disentangled from ϵ due to the suppression of $\mathcal{P}_{\zeta\phi}$. Large \hat{r} beyond the plotted range further suppresses $\mathcal{P}_{\zeta\phi}$, until errors discussed below (23) become important. For this case with constant potential tilts, one can compute the $|\dot{H}/H^2|$ effects by solving the field dynamics and check that $\partial\mathcal{N}/\partial\phi_*$ (22) further receives a contribution of $\sim \epsilon\mathcal{N}_a \times H_*/\dot{\phi}_*$ when $\hat{r} \gg 1$.

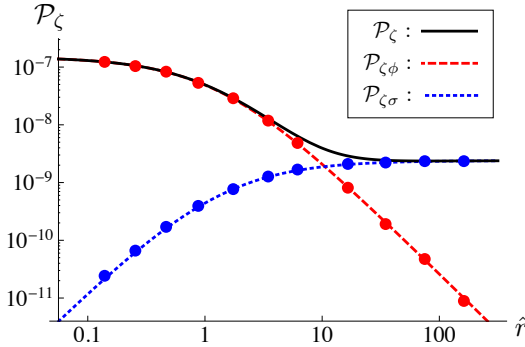


FIG. 2: Amplitude of density perturbations as a function of the energy density ratio \hat{r} . Inflaton-induced perturbations are suppressed at $\hat{r} \gg 1$.

Conclusions.— The density perturbations seeded by the inflaton can be suppressed when having additional light fields. This has important implications for inflationary cosmology, especially when relating inflaton field excursions with the primordial gravitational waves. Generalized forms of the Lyth bound (11) and (12) were also derived. The results can further be extended to general multi-field inflation models. We also note that the Lyth bound may be completely evaded when having dynamical fields whose super-horizon fluctuations themselves are suppressed, which may happen in some rapid-roll inflation models or heavy oscillating curvaton scenarios.

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- [11] The total field excursion would not simply be (11) multiplied by the total e-folding number if $d\phi/d\mathcal{N}$ suddenly changes during inflation [7]. However, even in such cases, obtaining observably large r over the entire CMB scales from a single inflaton field requires $\Delta\phi$ of order the Planck scale, if not super-Planckian.
- [12] The inflaton-induced perturbations can also be suppressed in the conventional curvaton scenario where the oscillation starts after inflation. Though we remark that for a simple quadratic curvaton potential $V = m_\sigma^2 \sigma^2 / 2$, the parameter κ is suppressed by m_σ^2 / H_*^2 and is much smaller than unity. One also finds that in order to have $\kappa \approx 1$ from modulated reheating, then the time variation of the inflaton decay rate becomes rather large, requiring treatments beyond sudden-decay approximations [8].
- [13] One may wonder whether σ_{osc} is homogeneous since it may also depend on the Hubble parameter, as is seen in the condition for the onset of oscillation $|\dot{\sigma}/H|_{\text{osc}} \sim |\sigma_{\text{osc}} - \sigma_{\text{min}}|$ where σ_{min} is the potential minimum [9]. We note that since H is nearly constant during inflation, σ_{osc} can be treated as a homogeneous constant.
- [14] The curvaton may initially undergo nonsinusoidal oscillations (e.g. (24)), but this does not alter the main results.
- [15] For slow-roll inflation the field excursion is related to the time variation of H as $(\dot{\phi}/M_p H)^2 = -2\dot{H}/H^2$. Here we consider $|\dot{H}/H^2|$ during inflation to be very small and nearly constant.
- [16] The slow-roll approximation of σ and the neglect of ρ_σ during inflation can also induce errors. Furthermore, the behavior of the oscillating curvaton deviates from that of matter at time scales shorter than its oscillation period. Such effects give rise to corrections typically of $\sim H_{\text{inf}}/m_\sigma$ (m_σ : mass during oscillation).
- [17] Since we have used the approximation $H \simeq H_{\text{inf}}$, one should be careful when discussing scale-dependencies of the perturbations. In particular, one should not simply carry out a k -derivative of the far right hand side of (23) when computing the spectral index.
- [18] The initial oscillations along the linear part of the potential may lead to formation of oscillons. However we do not expect them to modify the main results since they behave as matter, and also because they should not affect perturbations at the CMB scales which have exited the horizon long before their formation.